

Lecture 6 (Feb 15, 2016)

- Norm
- A vector norm is a function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ with the following properties:
for any $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$
 - $\|x\| \geq 0$, with $\|x\| = 0$ iff $x=0$.
 - $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)
 - $\|\alpha x\| = |\alpha| \|x\|$, $\forall \alpha \in \mathbb{R}$

we consider the class of p -norms, defined by

$$\begin{cases} \|x\|_p = \left(|x_1|^p + \dots + |x_n|^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \|x\|_\infty = \max_i |x_i| \end{cases}$$

Most commonly used norms:

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} = (x^T x)^{\frac{1}{2}}, \quad \|x\|_\infty = \max_{i=1 \dots n} |x_i|$$

- All p -norms are equivalent: $\forall p & q, \exists c_1, c_2 > 0$ st.

$$c_1 \|x\|_p \leq \|x\|_q \leq c_2 \|x\|_p$$

$$\text{e.g. } \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2, \|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty, \|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$$

- Holder inequality $|x^T y| \leq \|x\|_p \|y\|_q, \frac{1}{p} + \frac{1}{q} = 1$

- A matrix norm is a natural extension of vector norms to matrices.

Induced or operator norm:

For any given norm $\|\cdot\|$ on \mathbb{R}^n , the induced or operator norm on $\mathbb{R}^{n \times n}$ corresponding to $\|\cdot\|$ is defined as follows. For any matrix $A \in \mathbb{R}^{n \times n}$

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

- The induced norm corresponding to p -norms:

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

- For the common 1-norm, 2-norm & ∞ -norm, the induced norms are as follows:

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|, \quad \|A\|_\infty = \max_{1 \leq i \leq \infty} \sum_{j=1}^\infty |a_{ij}|, \quad \|A\|_2 = \sqrt{\lambda_{\max} A^* A}$$

e.g. $A = \begin{pmatrix} 1 & -5 \\ -2 & 3 \end{pmatrix} \Rightarrow \|A\|_1 = \max \{ |1| + |-2|, |-5| + |3| \} = 8$
 $\|A\|_\infty = \max \{ |1| + |5|, |-2| + |3| \} = 6$

$$\left(\begin{pmatrix} 1 & -5 \\ -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ -5 & 3 \end{pmatrix} \right) = \begin{pmatrix} 26 & 17 \\ 17 & 13 \end{pmatrix} \quad \|A\|_2 = \sqrt{\lambda_{\max} \begin{pmatrix} 26 & 17 \\ 17 & 13 \end{pmatrix}} \approx \sqrt{37.7} = 6.14$$

matrix measure (Vidyasagar - chap 2)

Let $\|\cdot\|$ be an induced matrix norm on $\mathbb{R}^{n \times n}$. Then the corresponding matrix measure (logarithmic norm) is the function $M: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by

$$M(A) := \lim_{h \rightarrow 0^+} \frac{\|I+hA\|-1}{h}$$

The matrix measure can be thought of as the directional derivative of the induced norm $\|\cdot\|$, in the direction of A & evaluated at identity matrix I .

- Matrix measure induced by p -norms denoted by M_p and for common p -norms, $p=1, 2, \infty$, can be written as follows:

$$M_1(A) = \max_{j=1 \dots n} \sum_{\substack{i=1 \\ i \neq j}}^n |a_{ii}| + |a_{ij}| \quad M_2(A) = \lambda_{\max} \frac{A + A^T}{2}$$

$$M_\infty(A) = \max_{i=1 \dots n} \sum_{\substack{j=1 \\ i \neq j}}^n |a_{ji}| + |a_{ij}| \quad \text{Example. } A = \begin{pmatrix} -4 & 0 \\ -2 & -3 \end{pmatrix}$$

Fundamental properties of solutions of ODEs (chapter 3)

$$\dot{x} = f(t, x), x(t_0) = x_0 \quad x \in \mathbb{R}^n, t \in [t_0, \infty)$$

1) Does a solution exist? If so, is it a unique solution?

Key: Lipschitz condition on $f(t, x)$.

2) Continuous dependence of solutions on initial condition (t_0, x_0) and on parameters of f .

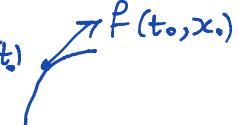
Existence & Uniqueness (sec 3.1)

Suppose $f(t, x)$ is continuous in both t and x .

Then solutions always exist and $x(t)$ is continuously differentiable.

Idea: given any (tangent) vector in \mathbb{R}^n , at least one curve exists s.t.

the vector is tangent to it:



However, can be more than one solution:

Example

$$\begin{aligned} \dot{x} = x^{1/3} &\rightarrow \int_{x_0}^{x(t)} \frac{dx}{x^{1/3}} = \int_{t_0}^t dt \rightarrow \left. \frac{3}{2} x^{2/3} \right|_{x_0}^{x(t)} = t - t_0 \\ &\rightarrow \frac{3}{2} (x(t)^{2/3} - x_0^{2/3}) = t - t_0 \end{aligned}$$

when $t_0 = 0, x_0 = 0$, then $x(t) = (2/3 t)^{3/2}$

But $\boxed{x(t)=0}$ is also a solution.

f is continuous \Rightarrow existence ✓
uniqueness ✗

Local Existence & Uniqueness

Def $f(t, x)$ is "piecewise continuous" in t on an interval $J \subset \mathbb{R}$ if for any bounded J_0 , $J_0 \subset J$, f is continuous in t , $\forall t \in J_0$, except possibly at a finite number of points where f may have finite-jump discontinuity

Def $f(t, x)$ is "locally Lipschitz in x at x_0 ", if there exists a neighborhood $B(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$ where $f(t, x)$ satisfies the "Lipschitz condition":

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|, \quad L = L(x_0) > 0$$

$\forall x, y \in B(x_0, r)$ & $t \in [t_0, t_1]$ the interval where $f(\cdot, x)$ is defined.

Def. $f(t, x)$ is "locally Lipschitz in x on a domain D " ($D \subseteq \mathbb{R}^n$ open & connected) if it is locally Lipschitz at every point $x_0 \in D$, with Lip. const $L = L(x_0)$.

How to check for Lipschitz condition?

- By definition

- By computing $\frac{\partial f}{\partial x}(x, t)$ (see below)

- When $n=1$ & f depends only on x , Lipschitz condition is

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L$$

i.e. on a plot of $f(x)$ versus x , a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L .

→ Any function that has infinite slope at some point is not locally Lipschitz at that point. e.g. $x = x^{1/3}$ 

→ A discontinuous function is not locally Lipschitz at the point of dis. cont

→ On the other hand, if $f'(x)$ is continuous at a point x_0 , then $f(x)$ is locally Lip. at that point: $f: \text{cont at } x_0 \Rightarrow \|f'(x_0)\|: \text{bounded on a neighborhood of } x_0$ ($|f'(x)| \leq K, \forall x \in B(x_0, r)$) $\Rightarrow f$ satisfies Lip. condition for $L = K$.

more generally, we have the following lemma:

Lemma 3.1 Let $f: [t_0, t_1] \times D \rightarrow \mathbb{R}^m$ be continuous for some $D \subseteq \mathbb{R}^n$.

If $\frac{\partial f}{\partial x}$ exists & is continuous & "uniformly bounded" on a convex set W , i.e.
 $\left\| \frac{\partial f}{\partial x}(t, x) \right\|_p \leq L$ (L does not depend on t, x). Then

$$\|f(t_1, x) - f(t_0, y)\|_p \leq L \|x - y\|_p \quad \forall t \in [t_0, t_1] \quad \& \quad x, y \in W$$

Proof. By mean value theorem

Fix $t \in [t_0, t_1]$ & $x, y \in W$.



W convex $\rightarrow y(s) := (1-s)x + sy \in W$ for $s \in [0, 1]$

Take $z \in \mathbb{R}^m$ st $\|z\|_q = 1$ & $z^T (f(t_1, y) - f(t_0, x)) = \|f(t_1, y) - f(t_0, x)\|_p$

$\exists z$ (see Ex 3.21) (where $\frac{1}{p} + \frac{1}{q} = 1$)

Define $g(s) := z^T f(t, y(s)) : [0, 1] \rightarrow \mathbb{R}$, c'

$\xrightarrow[\text{for } g]{\text{mean value thm}} g(1) - g(0) = g'(s_1) \text{ for some } s_1 \in (0, 1)$

$$\Rightarrow z^T (f(t_1, y) - f(t_0, x)) = g(1) - g(0) = g'(s_1) = z^T \frac{\partial f}{\partial x}(t, y(s_1)) (y - x) \quad \text{chain rule}$$

$$\xrightarrow{\text{Hölder}} |z^T (f(t_1, y) - f(t_0, x))| \leq \|z\|_q \left\| \frac{\partial f}{\partial x}(t, y(s_1)) \right\|_p \|y - x\|_p$$

$$\Rightarrow \|f(t_1, y) - f(t_0, x)\|_p \leq L \|y - x\|_p$$

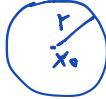
Lemma 3.2. For $t \in [t_0, t_1]$, $x \in D \subseteq \mathbb{R}^n$, $f(t, x)$ & $\frac{\partial f}{\partial x}$: cont. $\Rightarrow f$ locally Lip on D .

Proof. $\forall x_0 \in D$, find a neighborhood $B(x_0, r) \subseteq D$ st. $\overline{B(x_0, r)} \subset D$. Since $\overline{B(x_0, r)}$ is compact & convex, $\left| \frac{\partial f}{\partial x} \right| \leq L$ on $[t_0, t_1] \times \overline{B(x_0, r)}$ $\xrightarrow[3.1]{\text{Lemma}} f$: locally Lip.

Theorem 3.1.

Let $f(t, x)$ be piecewise continuous in t & satisfy Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|$$



$\forall x, y \in B_r(x_0) \quad \forall t \in [t_0, t_1]$

Then $\exists \delta > 0$ st. $\dot{x} = f(t, x)$, $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

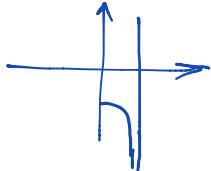
□ Without local Lip. condition, we cannot ensure uniqueness. e.g. $\dot{x} = x^{1/3}$.
 $x(0) = 0$.

□ The theorem is a "local" result: The interval $[t_0, t_0 + \delta]$ might not include the given interval $[t_0, t_1]$. Indeed the solution may cease to exist after some time:

Example. (recall) $\dot{x} = -x^2$, $x(0) = -1$

$f(x) = -x^2$ is locally Lipschitz for all x .

$$x(t) = \frac{1}{t-1} \xrightarrow[t \rightarrow 1^-]{} -\infty$$



In general: If $f(t, x)$ is locally Lip. on D and the solution of $\dot{x} = f(t, x)$ has a finite escape time t_e , then the solution $x(t)$ must leave every compact (closed & bounded) subset of D as $t \rightarrow t_e$.

Global Existence and Uniqueness

Def $f(t, x)$ is globally Lip. in x if for all $x, y \in \mathbb{R}^n$

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\| \quad \text{with the same Lip. const. } L.$$

The following lemma (which is a generalization of Lemma 3.1), gives a tool to check the global Lip. condition.

Lemma 3.3. f & its partial derivatives $\frac{\partial f_i}{\partial x_j}$ are continuous for all $x \in \mathbb{R}^n$, then

$f(t, x)$ is globally Lip. in x iff $\frac{\partial f_i}{\partial x_j}$ are globally bounded uniformly in t :

$$\left\| \frac{\partial f_i}{\partial x_j}(x, t) \right\| \leq L \quad \forall x, t$$

Example. $f(x) = -x^2$: locally Lip but not globally Lip because

$f'(x) = -2x$ is not globally bounded.

Theorem 3.2.

Let $f(t, x)$ be piecewise cont. in t & globally Lip. in x , $\forall t \in [t_0, t_1]$. Then $\dot{x} = f(t, x)$, $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$.

→ The global Lipschitz condition is satisfied for linear systems of the form $\dot{x} = A(t)x + g(t)$. But it is a restrictive condition for general non-linear systems.

However, there exist vector fields that do not meet the conditions of the theorem but have well-defined flows:

Example. $\dot{x} = -x^3 = f(x)$

$f(x)$ is not globally Lip. because $f'(x) = -3x^2$ is not globally bounded.

The solution from $x(t_0) = x_0$ is $x(t) = \text{sign } x_0 \sqrt{\frac{x_0^2}{1 + 2x_0^2(t-t_0)}}$

Theorem. Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x , for all $t \geq t_0$ and all x in $D \subseteq \mathbb{R}^n$. Let $W \subseteq D$ be compact and suppose that every solution of $\dot{x} = f(t, x)$, $x(t_0) = x_0$ with $x_0 \in W$ remains entirely in W . Then there exists a unique solution defined for all $t \geq t_0$.

Back to $\dot{x} = -x^3$:

$$x(t) > 0 \rightarrow \dot{x}(t) < 0 \quad \& \quad x(t) < 0 \rightarrow \dot{x}(t) > 0$$

Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in \mathbb{R}, |x| \leq |a|\}$.

\Rightarrow The equation has a unique solution for all $t \geq 0$.

